Journal of Global Optimization **29:** 121–123, 2004.

A LETTER TO THE EDITOR

A Comment on: F.K. Hwang, Y.M. Wang and J.S. Lee, 'Sortability of Multi-Partitions', Journal of Global Optimization 24 (2002), pp. 463–472

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(Received 3 September 2003; accepted 14 October 2003)

Abstract. We correct a claim, made in [6], that the proof of the key result in [3] about the existence of a monotone optimal multi-partition was incomplete; further, we provide a sortability interpretation for the criticized step of that proof.

Consistency and sortability of properties of combinatorial objects provide a framework for generating objects that satisfy a prescribed property within a given class of objects, by iteratively moving within the class by making 'local changes'. More specifically, consider a family F of combinatorial objects, a property Q that elements of F may have, and a statistics S over the elements of F. If S is guaranteed to decline under 'local changes' of elelments in F that are aimed at acquiring Q in a 'local sense' while maintaining the affiliation in F, then it is possible to use an iterative process that will generate an element in F which 'satisfies Q locally'. If property Q is consistent, that is, 'local satisfiability' suffices for 'global satisfiability', then such processes will generate objects in F that are guaranteed to satisfy Q. Sortability and consistency were introduced in the context of partitions over a one-dimensional parameter spaces in [4] (see [1] for a comprehensive study within this context). The approach was extended to partitions over multi-dimensional parameter spaces in [2] and to multi-partitions in [6].

Suppose that t and p are positive integers, and t types of items are to be assigned to p parts subject to prescribed specifications. Specifically, there are nonnegative integers $\{n_{uj}:u=1,\ldots,t,j=1,\ldots,p\}$. For each $u=1,\ldots,t, n_u\equiv\sum_{j=1}^{p}=n_{uj}$ items, indexed $1,\ldots,n_u$, of type u are to be assigned to parts indexed $1,\ldots,t$, with n_{uj} items of type u assigned to part j. A *multi-partition* is such an assignment; formally, a multi-partition π is a collection $\{\pi_{uj}:u=1,\ldots,t,j=1,\ldots,p\}$ where for each $j,\{\pi_{uj}:j=1,\ldots,p\}$ partitions $\{1,\ldots,n_u\}$ and for each u and $j,|\pi_{uj}|=n_{uj}$. A multi-partition π is *monotone* if there exists a ranking $j_1,\ldots,j_p$ of $\{1,\ldots,p\}$ such that part j_1 gets the first n_{uj_1} items of each type u, part j_2 gets the next n_{uj_2} items of each type u, and so on, until part j_p get the last n_{uj_p} items of each type u, that is, if $\pi = \{\pi_{uj} : u = 1, \ldots, t, j = 1, \ldots, p\}$, then $\pi_{uj_s} = \{\sum_{r=1}^{s-1} n_{uj_r} + 1, \ldots, \sum_{r=1}^{s} n_{uj_r}\}$ for each $u = 1, \ldots, t$ and $s = 1, \ldots, p$.

Suppose that each item i of type u is associated with a real number $0 < r_{ui} \leqslant 1$; without loss of generality we will assume that items are numbered so that

$$0 < r_{ul} \leqslant r_{u2} \leqslant \cdots \leqslant r_{un_u} \leqslant 1 \quad \text{for each } u = 1, \ldots, t. \tag{1}$$

Also, assume that a monotone (Boolean) function J: $\{0,1\}^P \to \{0,1\}$ is given. For a multi-partition $\pi = \{\pi_{uj}$: $u = 1, \ldots, t,\ j = 1, \ldots, p\}$ let

$$r(\pi)_j \equiv \prod_{u=1}^{t} \prod_{j \in \pi_{uj}} r_{ui} \quad \text{for each } j = 1, \ldots, p.$$

and

$$R(\pi) \equiv \sum_{s \in \{0,1\}^P} J(s) \left\{ \prod_{\{j:sj=0\}} [1 - r(\pi)_j] \right\} \left\{ \prod_{\{j:sj=1\}} r(\pi)_j \right\}.$$

It is claimed in [6] that the proof in [3] that there exists a monotone multi-partition π which maximizes $R(.)$ over all multi-partitions has a flaw as the statistics $\sum_j \sum_u \max \pi_{uj}$ need not decline strictly as pairs of parts are rearranged iteratively (while preserving optimality). But, the argument in [3] made no such claim. The core of the argument in [3] lies in an an inductive proof (Lemma 2) that considers a relaxation of the problem (removed later on) that requires that the inequalities in (1) hold strictly and that some item type w has $n_{wj} > 0$ for each j. The adopted inductive hypothesis is that some parts $j_1, \ldots, j_{k-1}$ are ordered monotonically, in order, in some optimal multi-partition π (that is, the parts in j_1 of all types are those with the smalles indices, those in j_2 are the next ones, etc.). Let j_k be the (unique) index for which π_{wj_k} contains $\sum_{r=1}^{k-1} n_{wj_r} + 1$. With π' as the multi-partition that minimizes $[\sum_u \max \pi_{uj_k}]$ over the optimal multi-partitions τ that have parts $j_1, \ldots, j_{k-1}$ fixed in the above monotone arrangement and have $\sum_{r=1}^{k-1} n_{wj_r} + 1 \in \tau_{wj_k}$, it is argued that π' and $j_1, \ldots, j_k$ satisfy the inductive hypothesis with k repalacing $k-1$. With $k = p$, the inductive hypothesis establishes that there exists a monotone optimal multi-partition. Essentially, the argument we just described shows that within the class of multi-partitions τ that have parts $j_1, \ldots, j_{k-1}$ fixed in a monotone arrangement and have $\sum_{r=1}^{k-1} n_{wj_r} + 1 \in \tau_{wj_k}$, one can rearrange pairs of parts so as to reduce $\sum_u \max \pi_{uj_k}$. But, no claim is made in [3] about reducing $\sum_j \sum_u \max \pi_{uj}$; in fact, examples in [6, Section 5] demonstrate that $\sum_j \sum_u \max \pi_{uj}$ need not decline in the above process.

The approach followed in [3] (and explained in the preceding paragraph) can be cast as a sortability argument with the *lexicographic* statistics:

$$S(\pi) \equiv \left(\min_{j_1} \frac{\sum_u \max \pi_{uj_1}}{\sum_u n_{uj_1}}, \min_{j_1,j_2} \frac{\sum_u \max \pi_{uj_1} + \sum_u \max \pi_{uj_2}}{\sum_u n_{uj_1} + \sum_u n_{uj_2}}, \ldots, \right.$$
$$\left. \min_{j_1,\ldots,j_k} \frac{\sum_{r=1}^{k} \sum_u \max \pi_{uj_r}}{\sum_{r=1}^{k} \sum_u n_{uj_r}}, \ldots, \min_{j_1,\ldots,j_p} \frac{\sum_{r=1}^{p} \sum_u \max \pi_{uj_r}}{\sum_{r=1}^{p} \sum_u n_{uj_r}} \right).$$

where maxima over empty sets are defined as 0 and the assumption $n_{wj} > 0$ for each j assures that no denominator is zero. In particular, note that $S(\pi) = (1, \ldots, 1)$ if and only if π is monotone. The essence of the proof in [3] is the demonstration that, when the inequalities in (1) are strict, $S(.)$ can be reduced (lexicographically) within the class of optimal multi-partitions which are not monotone by pairwise shuffling of parts. This is accomplished by using the fact [3, Corollary 1] that if there is an optimal multi-partition which is not (j_1, j_2)-monotone, then all partitions which are the result of shuffles among part j_1 and j_2 are optimal. This property depends on the convexity of the function g: $(-\infty, 0]^p \to (0, 1]$ with

$$g(y) = \sum_{s \in \{0,1\}^p} J(s) \left[\prod_{\{i:s_i=0\}} (1 - e^{y_i}) \right] \left[\prod_{\{i:s_i=1\}} e^{y_i} \right] \quad \text{for every } y \in (-\infty, 0]^p,$$

and does not extend to asymmetric Schur convex functions (see [5]). We further note that the removal of the assumptions that one item-type is required by all the parts and that the inequalities of (1) hold strictly are not standard 'sortability arguments' (as considered, for example, in [6]).

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